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An algorithm for presenting pairs in optimum orders

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Summary

In 1934, R. T. Ross published tables for presenting pairs of stimuli in an optimum order. Ross's optimum order was characterized by two specifications:

a) maximum spacing: the number of pairs which appear between a first occurrence and every subsequent occurrence of the same pair element ist maximised

b) balance: an element in a pair appears as often on the right as on the left.

Zusammenfassung

Der Psychometriker R. T. Ross publizierte im Jahr 1934 eine Reihe von Tabellen für die optimale Anordnung von Paaren als Grundlage beim paarweisen Vergleich. Ross' optimale Anordnung, die zunehmend in der Praxis des Kriterienvergleichs angewendet wird, stützt sich auf zwei Maßstäbe:

a) maximaler Abstand: die Zahl der Paare zwischen einem ersten und jedem folgenden Erscheinen eines bestimmten Paar-Elementes sollte möglichst groß sein

b) Gleichgewicht: Jedes Paar-Element erscheint mit gleicher Häufigkeit auf der linken und rechten Seite.
Die vorliegende Arbeit beschreibt einen Algorithmus für die optimale Anordnung von Paaren nach Ross' Muster. Der Algorithmus, der auf der Graphentheorie basiert, läßt sich relativ einfach programmieren.

Ross' table are frequently being used in practical applications of pairwise comparisons. This article presents an algorithm for constructing Ross' oder in terms of complete graphs. The algorithm can be easily implemented on a computer.

Introduction

The assessment of judgmental information belongs traditionally to a branch of Psychology which is known as Psychometrics or Psychophysics. One of the most well-known Psychometric "laws" is Thurstone's Law of Comperative Judgement which is based on the assumption that relative preferences among different criteria ("stimuli") can be established if the criteria are lined up in pairs. One of the scientists who made an important contribution to the method of presenting pairs of stimuli in an optimum order was ROSS (1934, 1939). The method proposed by ROSS which is also being used in forestry (vide HULL, BUHYOFF and DANIEL, 1984), has the following characteristics:

1. An alternative is compared with every other alternative exactly once.
2. The pairs of alternatives are presented in a specific order. This order maximizes the number of pairs which appear between a first occurence und every subsequent occurence of the same alternative (ROSS' maximum spacing).
3. ROSS defines a characteristic, called balance, such that an alternative in a pair appears as often on the right as on the left.

ROSS did not give a formal mathematical development of his method. We shall construct ROSS' order in terms of complete graphs (CLOETE & CLOETE, in press).

An algorithm

First it is necessary to define a few graph-theoretic concepts. A complete graphs K_n is an ordered triple (V, E, ψ) consisting of a set V of vertices, E of edges (disjoint from V) and an incidence function ψ that associates with each edge of K_n two distinct vertices, such that there is an edge associated with every two distinct vertices (BONDY and MURTY, 1978). K_n has n vertices and thus $n(n-1)/2$ edges. We only consider the case where n is odd and $n \geq 5$. See figure 1 for a K_5 graph. Two edges which are incident with a common vertex are adjacent. A k-edge colouring of K_n is an assignment of k colours, $1, 2, \ldots, k$, to the edges of K_n. The colouring is proper if not two adjacent edges have the same colour. A proper k-edge colouring is a partition $(E_1, E_2, \ldots, E_k)$ of E, where E_i is a subset of E assigned colour i. The smallest number for which K_n has a proper edgecolouring is n.

Let an edge (v_1, v_2), where v_1 and v_2 are its vertices, represent a pair of alternatives v_1 and v_2. Owing to characteristic 1 it is clear that a complete graph is required when each pair in the ROSS order is represented by an unique edge.

In order to obtain maximum spacing (characteristic 2), find a maximum number of non-adjacent edges of K_n. That is, find subsets E_i of E with a maximum number of edges. This occurs in a proper n-edge colouring of K_n, where every E_i, $i = 1, \ldots, n$, has $(n-1)/2$ elements (BERGE, 1957). Thus maximum spacing is obtained within each E_i. Note that the E_i are

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Table 1. Table used for finding a proper n-edge colouring; for details see text

Column	1	2	3	$m-1$	m
E_1	$(m+1, m+3)$	$(m, m+4)$	$(m-1, m+5)$	$(3, n)$	$(2, 1)$
E_3	$(m+2, m+3)$	$(m+1, m+5)$	$(m, m+6)$	$(4, 2)$	$(3, 1)$
E_5	$(m+3, m+4)$	$(m+2, m+5)$	$(m+1, m+7)$	$(5, 3)$	$(4, 1)$
E_7	$(m+4, m+5)$	$(m+3, m+6)$	$(m+2, m+7)$	$(6, 4)$	$(5, 1)$
E_{n-6}	$(n-3, n-2)$	$(n-4, n-1)$	$(n-5, n)$	$\left(\frac{n-1}{2}, \frac{n-5}{2}\right)$	$\left(\frac{n-3}{2}, 1\right)$
E_{n-4}	$(n-2, n-1)$	$(n-3, n)$	$(n-4, 2)$	$\left(\frac{n+1}{2}, \frac{n-3}{2}\right)$	$\left(\frac{n-1}{2}, 1\right)$
E_{n-2}	$(n-1, n)$	$(n-2, 2)$	$(n-3, 3)$	$\left(\frac{n+3}{2}, \frac{n-3}{2}\right)$	$\left(\frac{n+1}{2}, 1\right)$
E_n	$(n, 2)$	$(n-1, 3)$	$(n-2, 4)$	$\left(\frac{n+5}{2}, \frac{n-1}{2}\right)$	$\left(\frac{n+3}{2}, \frac{n+1}{2}\right)$
E_2	$(2, 3)$	$(n, 4)$	$(n-1, 5)$	$\left(\frac{n+7}{2}, \frac{n+1}{2}\right)$	$\left(\frac{n+3}{2}, 1\right)$
E_4	$(3, 4)$	$(2, 5)$	$(n, 6)$	$\left(\frac{n+7}{2}, \frac{n+3}{2}\right)$	$\left(\frac{n+5}{2}, 1\right)$
E_6	$(4, 5)$	$(3, 6)$	$(2, 7)$	$\left(\frac{n+9}{2}, \frac{n+5}{2}\right)$	$\left(\frac{n+7}{2}, 1\right)$
E_8	$(5, 6)$	$(4, 7)$	$(3, 8)$	$\left(\frac{n+11}{2}, \frac{n+7}{2}\right)$	$\left(\frac{n+9}{2}, 1\right)$
E_{n-7}	$(m-2, m-1)$	$(m-3, m)$	$(m-4, m+1)$	$(n-2, n-4)$	$(n-3, 1)$
E_{n-5}	$(m-1, m)$	$(m-2, m+1)$	$(m-4, m+2)$	$(n-1, n-3)$	$(n-2, 1)$
E_{n-3}	$(m, m+1)$	$(m-2, m+2)$	$(m-3, m+3)$	$(n, n-2)$	$(n-1, 1)$
E_{n-1}	$(m, m+2)$	$(m-1, m+3)$	$(m-2, m+4)$	$(2, n-1)$	$(n, 1)$

ordered sets. However, an ordered sequence of the E_i such that maximum spacing is also maintained in the overall order, is required. For an E_i, $i = 1, \ldots, n$, there is exactly one vertex, say v_i, where $v_i \notin V_i$, the vertex-set of E_i, but $v_i \in V_j$, $\forall j \neq i$. To construct an overall order with maximum spacing this v_i is used as a vertex of the first edge of E_{i+1}, $i = 1, \ldots, n-1$.

An algorithm to construct the Ross order is presented. Let $m = (n-1)/2$. We label the vertices of K_n using the numbers $1, \ldots, n$. The algorithm consists of three steps:

(a) Find a proper n-edge colouring by means of table 1. The rows of table 1 give the subsets E_i of the partition. Each entry is an edge (v_i, v_j) of K_n. Construct the partition using table 1 as follows:

1. Select the first m-1 columns as well as the last column, and the first m rows, the middle row (with colour n) as well as the last m rows. This defines an n by m matrix of edges.
2. Now consider the first m-1 columns and all n rows, and for every vertex

 if $v_i > n$ then let $v_i = (1+v_i) \bmod_n$ or
 if $v_i < 2$ then let $v_i = n-1+v_i$.

(b) Pack the resulting edges to obtain the final ordering by means of table 2 which contains the position number of the corresponding edge found in table 1.

(c) Balance the vertices to satisfy characteristic 3. Note for n odd it is always possible to achieve perfect balance (CLOETE & CLOETE, in press).

A computer algorithm is implemented in Pascal (JENSEN and WIRTH, 1974)*.

It is clear that the ROSS order for any n is not unique because K_n has a number of proper n-edge colourings. For K_5, for example, we can give three orders which satisfy the characteristic of maximum spacing:

(2,1), (5,3), (1,4), (3,2), (4,5), (1,3), (2,5), (3,4), (5,1), (4,2);
(2,1), (3,5), (4,2), (1,3), (5,4), (3,2), (5,1), (4,3), (2,5), (1,4);
(1,2), (5,3), (4,1), (2,5), (3,4), (1,5), (4,2), (3,1), (5,4), (2,3).

Note that there are proper n-edge colourings which do not satisfy the characteristic of maximum spacing.

An example

Suppose we are given the task to set the priorities in a silvicultural research programme, involving the following 5 major fields of research:

Forest Ecology, Soils & Climate	(1)
Species Trials and Tree Breeding	(2)
Applied Silviculture	(3)
Forest Protection	(4)
Scientific Services (Biometry)	(5)

Table 2. Table used for packing edges; for details see text

	Column					
	1	2	3	$m-2$	$m-1$	m
E_1	m	$m-1$	$m-2$	3	2	1
E_3	1	m	$m-1$	4	3	2
E_5	2	1	m	5	4	3
E_{n-4}	$m-2$	$m-3$	$m-4$	1	m	$m-1$
E_{n-2}	$m-1$	$m-2$	$m-3$	2	1	m
E_n	m	$m-1$	$m-2$	3	2	1
E_2	2	3	4	$m-1$	m	1
E_4	3	4	5	m	1	2
E_6	4	5	6	1	2	3
E_{n-5}	$m-1$	m	1	$m-4$	$m-3$	$m-2$
E_{n-3}	m	1	2	$m-3$	$m-2$	$m-1$
E_{n-1}	1	2	3	$m-2$	$m-1$	m

* Information about the programme by I. CLOETE

The 5 criteria are lined up in $n(n-1)/2 = 10$ Pairs, which are represented by the K_5 graph. Every E_i consists of 2 edges (with the same colour) and there are 5 ordered coloursets, given by

$$E_1 = \{ (1,2), (5,3) \}$$
$$E_2 = \{ (4,1), (3,2) \}$$
$$E_3 = \{ (4,5), (1,3) \}$$
$$E_4 = \{ (2,4), (5,1) \}$$
$$E_5 = \{ (3,4), (2,5) \}$$

which is the ROSS pairing for $n = 5$:

(1,2), (5,3), (4,1), (3,2), (4,5),
(1,3), (2,4), (5,1), (3,4), (2,5)

obtained from the computer algorithm. The resulting colouring is illustrated in figure 1.

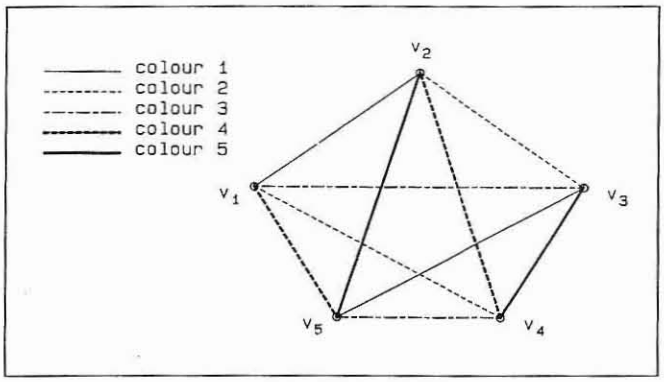

Fig. 1. Edge-colouring for $n = 5$

The set of pairs is presented in Table 3.

Standard methods are available for evaluating a particular assessment (VON GADOW, 1986), but we will not discuss these here.

Table 3. Example of a set of balanced pairs and completed assessment

How do you rate the relative importance of X against Y?

X	Assessment			Y
	more important	equally important	less important	
ecology		*		breeding
services	*			applied
protection		*		ecology
applied	*			breeding
protection			*	services
ecology		*		applied
breeding	*			protection
services	*			ecology
applied		*		protection
breeding	*			services

Remarks

Many decisions are taken on the basis of foresters' personal perception of things. Mathematical models for the description of mental phenomena can be used to capture personal preferences.

The method of paired comparisons, using ROSS' optimal ordering of pairs, is a useful procedure for presenting criteria for assessment. The graph-analytic algorithm proposed in this paper can be easily implemented on a computer and this is convenient, especially if n is large. There is mathematical proof that ROSS' empirical spacings are optimal, though not uniquely optimal.

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