

DETAILED NON-LINEAR ASYMPTOTIC REGRESSION STUDIES ON
TREE AND STAND GROWTH WITH PARTICULAR REFERENCE TO
FOREST YIELD RESEARCH IN BAVARIA (FEDERAL REPUBLIC
OF GERMANY) AND INDIA*

By

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SUMMARY

The generalised growth function developed by RICHARDS (1959) has been found most suitable for the model construction of site index curve system. The existing methods of construction of site index curves, which constitute the most important part of the Normal Yield Tables, are proposed to be replaced by the model. BRICKELL (1968), LUNDGREN & DOLID (1970), BECK (1971), HÄGGLUND (1972) and RAWAT (1973) have successfully used the function for construction of site index curves with sufficient data from different sources. The methodology, however, seems different from author to author. Standard method of STEVENS (1951) for asymptotic regression seems to be most appropriate for construction of the model. HIORNS (1965) claims so already. He, however, does not mention the name of MARQUARDT (1963) therein. We feel that Marquardt's method is also suitable for any non-linear asymptotic function, also for the generalised growth function of Richards, but since a specific methodology has been developed by Stevens for Richards's growth function, the latter should be preferred. The biologists (foresters) can easily grasp the procedure.

The model is equally suitable for all other growth and yield studies, e.g. diameter/age, basal area/age, total volume/age and number of stems per unit area/age relationships. TURNBULL & PIENAAR (1965) constructed a model, also based on the growth function of Richards, to define the

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expected trend of growth for stands of different stocking levels taking basal area per acre over age as mathematical basis. There is no doubt that basal area is the simplest and most widely accepted measure of stocking density, but it should be expressed mathematically, if required, correctly and sufficiently. We have found that stand basal area per acre/age curves are not asymptotic in all the cases. These growth curves bend downwards after reaching the maximum value for stands of some of the species like Pine (*Pinus silvestris*, L.). Probably it is because of the fact that the site is not capable of supporting optimum number of stems at higher ages. The subject as such pertains to thinning research and detailed investigations and analyses are necessary in this direction. We have, however, analysed some data and found that stand basal area of mean tree/age and the number of stems per unit area/age, when combined, are better suited for defining different stocking levels of stands.

THOMASIU (1964) has used similar function for the site quality curves of Spruce (*Picea abies*, K.). The function, however, has been fitted to the data for individual site qualities and does not serve the purpose of a mathematical model in strict sense of the term. Similarly he has used so-called Korsun's function for the number of stems per hectare/age relationship, which is entirely empirical, see KORSUN (1935). The present paper, on the other hand, suggests inclusion of a standardised mathematical (growth) model in forestry research relating to growth of single trees and evenaged stands. The model is a growth model as it is a mathematical expression satisfying the general principles of biological growth. On the whole a concept of computerised yield tables is offered by the model without losing the desired accuracy. The model is given in summarised form as under:

$$\text{Growth Model --- } W = A \cdot (1 - b \cdot e^{-k \cdot T})^{\frac{1}{1-m}} \text{ -----(1)}$$

$$\text{where } A = a_0 + a_1 \cdot S + a_2 \cdot S^2 + a_3 \cdot S^3 + a_4 \cdot S^4 \text{ -----(2)}$$

$$b = b_0 + b_1 \cdot S + b_2 \cdot S^2 + b_3 \cdot S^3 + b_4 \cdot S^4 \text{ -----(3)}$$

$$k = k_0 + k_1 \cdot S + k_2 \cdot S^2 + k_3 \cdot S^3 + k_4 \cdot S^4 \text{ -----(4)}$$

$$m = m_0 + m_1 \cdot S + m_2 \cdot S^2 + m_3 \cdot S^3 + m_4 \cdot S^4 \text{ -----(5)}$$

$$\text{and } S = (H100 - e_0) / e_1 \text{ -----(6)}$$

In the equations W is size or growth dimension of a tree or evenaged stand, T is the age, A , b , k , m are the four growth parameters, S is the site quality, $H100$ is the site index at selected appropriate age, a_0 , a_1 , a_2 , a_3 , a_4 , b_0 , b_1 , b_2 , b_3 , b_4 , k_0 , k_1 , k_2 , k_3 , k_4 , m_0 , m_1 , m_2 , m_3 , m_4 , e_0 and e_1 are coefficients on which the growth parameters, site quality and site index are based.

Matrix operations have been kept to the minimum so as to make the model simple and understandable by the average forester. The model has been fitted to the data taken from the existing yield tables of various tree species in Bavaria (Germany) and India (Forest Research Institute, D'dun).

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INTRODUCTION

The site quality curves (site index curves) constitute the most important part of the yield tables and have been in use throughout the world for a very long time to determine the site quality of an evenaged stand. The forest research workers and foresters in various parts of the globe have continually attempted to develop techniques for the construction of these curves/tables, but so far no single technique can be called as perfect. In most of the silvicultural research codes, presently referring to GRIFFITH & PRASAD (1949), it is left to the choice of individual authors reasoning that it is not desirable to lay down any stereotyped method for compiling the sample plot data into yield tables. The authors in this paper propose a standardised technique of construction of a mathematical (growth) model for all the yield and growth curves in general and for the site quality curves in particular.

BRIEF HISTORY OF CONSTRUCTION OF SITE INDEX CURVES

SPURR (1952) and many other authors have summarised the existing methods of construction of site index curves and described the causes as to why differences between site index curves and observed height growth patterns occur. Leaving aside the method described by ILVSSALO (1927), Finland and which is based on Cajander's theory of forest types, the following two methods have been generally adopted for the purpose:

Method I viz the strip method (Streifenverfahren) begins in 1877 and is associated with the name of v. Baur, Germany. In this method, after plotting height or top height over age for individual plot data, two limiting curves are drawn, one delimiting the upper extent of the data and the other, the lower border. The space between the two limiting curves is then divided into strips of equal breadth by harmonised curves to construct the final site index tables.

The method is outdated as there are serious basic objections in it. The chief objection is that the guiding curves are based upon the extremes of the data and not upon the mean data. Moreover the data in the extremes usually belong to few sample plots only. Another objection is the assumption that the form of the growth curves for the middle site classes is determined by the form of the growth curves for the best and poorest sites.

Method II is the method of harmonised site index curves. It was originally promulgated by Heyer in 1846 (Leitkurvenverfahren) and is essentially the method of permanent sample plots. In this method a single mean curve (guiding curve) of height or top height over age is drawn and at any convenient (index) age, after which the period of rapid height growth is completed and which should preferably be somewhat less than the most usual rotation age for the species, a convenient number of equal height intervals are selected to limit site classes. A set of curves through the points so obtained is drawn harmonically with the guiding curve. Each curve of the family has the same shape and other characteristics of the guiding curve, only differing in magnitude by a fixed percentage. Some of the authors have applied the principles of anamorphosis or alinement charts (nomograms) to the technique of construction of site index curves.

Several assumptions are made in the method. The main assumptions are:

- (1) That the data of sample plots adequately sample the range in site with each age-class and consequently, the scatter-diagram of height over age adequately indicates the shape of the curves.
- (2) That the effect of difference in site on height growth is relatively the same at all ages.
- (3) That the growth curves for all the sites have the same shape.

Since none of these assumptions is true, attempts have been made by the research workers to modify the technique from time to time. The first assumption can only be satisfied by choosing a consistent, efficient and sufficient sampling technique for the forest stands. Economics of these stands plays a vital role in it and it should be safely left to the discretion of the skilled forest managers. At the same time it is also possible that the stands of all ages on all sites are not growing. See SPURR (1952), page 312.

The problem posed by the second assumption that the site is correlated with age and thus affects height growth differently at various ages has been studied in some greater detail by some authors and to name one by BRUCE & SCHUMACHER (1950). The coefficient of variation* of the stand height for each of the predetermined age-classes is calculated and plotted over age to determine how much above or below the guiding curve should each individual site curve be. This aspect has been studied by CURTIS (1964), BRICKELL (1968) etc. and the attention of the readers is drawn to these references.

To construct a suitable mathematical (growth) model for the site index curves we are mainly concerned with the third assumption, wherein the question arises as to whether the shape of the growth curves is same for all sites. As quoted by BECK (1971) the site index curve is simply a growth curve for a given genetic entity under a set of environmental conditions. We thus come across as many diffe-

*Coeff. of var. = (standard deviation/arithm. mean). 100

rent curve shapes as there are combinations of trees and environmental conditions - an almost infinite number. Therefore for economic reasons we construct site index curves only for a certain tree species or a province or a variety thereof occurring in abundance and falling under a certain forest type, assuming that the environmental conditions are more or less the same for the whole area under the forest type. It is this aspect of different curve shapes for different sites that we discuss here.

It is now a well established fact that site index curves are essentially polymorphic, which vary in shape from one site to another. Anamorphic curves hitherto prepared are gradually being replaced by the polymorphic curves, where enough reliable data from permanent sample plots are available to support the fact. The general biological principles involved with the growth process of trees are satisfied by the polymorphic curves and they, therefore, are to be preferred for estimation of site index.

STATISTICAL BASIS FOR SITE INDEX CURVES

The site quality curves are generally constructed graphically for complete number of site qualities. The construction of harmonised curves is completely left on the personal skill of the research worker, which varies from person to person and thus a chance is given to introduce human bias in it. Moreover if a crop belongs to a fractional quality interpolation between the successive curves or tables becomes necessary. Such interpolations also are tedious and subject to human error. Particularly in the large scale forest inventories the curves are not of much use if they are not represented by a suitable mathematical model. There are measurements of many sample points in forest inventories and if for each sample point we are required to estimate site index manually with the help of the graphical curves, it may cost us much in terms of time, money and energy. It is, therefore, of utmost importance that we think deeply of constructing the curves on statistical basis. In our studies the criterion of best fit was the least sum of squares (least standard error of estimate) about the non-linear asymptotic function used by us for model construction. Moreover it is also possible to compute standard errors of various growth parameters involved in the function. While making polynomial regression studies 'F' test was simultaneously applied to check whether a polynomial of higher order was actually required or not. Whereas further details are available in the methodology given in the following pages, here we would like to impress that the field of mathematical statistics has developed very fast in the last few decades and it is finding an everincreasing place in all subjects. We would, therefore, be doing a big injustice to forest yield science, if we did not use statistical basis in model construction for growth curves.

GENERAL BIOLOGICAL PRINCIPLES INVOLVED IN GROWTH PROCESS OF TREES IN EVENAGED STANDS

ASSMANN (1970) pages 41-48 and many other authors have already discussed the subject and before the model is described at length, it would be proper here to describe the general biological principles in the growth process of trees in evenaged stands and then explain how they have been satisfied by the model. The Growth curves have in general the following two characteristics:

- (1) They are asymptotic with the straight line $W=A$, see Fig. 1(a) as the time (age) approaches to ∞ .
- (2) There is one point of inflexion on the curve at an age, which varies with species to species and with the site quality within the species. This means that the mean annual rate of growth (current annual increment) up to this age increases, becomes maximum at this age and falls down after this age till it slowly reaches to zero i.e. till the full maturity after which the stand or stem dies. If these two principles of growth process are satisfied by the model, such a model should be fairly accurate and solve one of our important problems in forest yield science, since such models can be constructed for all the growth curves, very much required in forestry practice.

CONSTRUCTION OF MATHEMATICAL (GROWTH) MODEL FOR GROWTH CURVES USING THE FLEXIBLE GROWTH FUNCTION DEVELOPED BY RICHARDS

It would not be out of place to trace here a brief history of various growth functions used in applied biology as the subject is closely related to forest yield science. A good account is available in the paper of RICHARDS (1959) and others. A good account is available from the times of PÜTTER (1920) and HUXLEY (1932) as mentioned by Richards and THOMAS (1964). Also refer to LUDWIG (1929). The growth function developed by Richards is an extension of work of BERTALANFFY (1941). The function, also known as Chapman-Richards function in America, has been very often used in biometrics and related fields and its utility is everincreasing with the advent of electronic age. The function is a generalisation of many of the then existing popular functions e.g.:

- (1) Monomolecular growth function - $W = A.(1 - e^{-k \cdot T})$
The growth rate equals to $k.(A - W)$.
- (2) Autocatalytic growth function - $W = A / (1 + b.e^{-k \cdot T})$
(also known as logistic func.)
The growth rate equals to $k.W.(A - W) / A$.
- (3) Gompertz growth function - $W = A.e^{-b.e^{-k \cdot T}}$
The growth rate equals to $k.W.\log_e(A/W)$.

The generalised function developed by Richards is as under:

$$\text{Growth rate equation - } \frac{dW}{dT} = n \cdot W^m - K \cdot W$$

$$\text{Growth function after integration- } W = A \cdot (1 - b \cdot e^{-k \cdot T})^{\frac{1}{1-m}}$$

where

W represents the size at time T and A its ultimate limiting value; k is known as 'rate constant', which determines the spread of the curve along the time axis. The growth parameters A, b, k & m in the growth function and n, K & m in the growth rate function have been interpreted by Richards lucidly and they are related to each other as under (also see Fig. 1(a) and 1(b)):

In the growth rate function dW/dT is the growth rate at time T. The maximum growth rate occurs at -

$$W = I = m \left(\frac{1}{1-m}\right) \cdot (n/K) \left(\frac{1}{1-m}\right) = A \cdot m \left(\frac{1}{1-m}\right)$$

In the growth function -

$$A = (n/K) \left(\frac{1}{1-m}\right) = \text{Asymptote (ultimate limiting size)}$$

$$k = K \cdot (1-m)$$

$$b = 1 - (W_0/A)^{1-m}, \text{ where } W=W_0 \text{ when } T=0.0$$

The growth function of Richards can be fitted to data accurately by method of STEVENS (1951). The equation dealt with by Stevens is of the following form:

$$y = \alpha + \beta \cdot \rho^x$$

The equation is a transformed form of monomolecular growth function described above and is also known as MITSCHERLICH's law (1948), when used for describing the response of yield of a crop to a fertiliser or somewhat comparable influence factor. About two decades ago the equation was very much celebrated in various fields of human knowledge. It was used directly or in transformed forms in physics (Newton's law of cooling), in agriculture and biology as described above, in economics and human sociology for graduation of life tables by actuaries and for describing population growth by demographers.

Attempts of deducing growth functions in forest science have been made in Europe and particularly in Germany from very old times. ASSMANN (1970) has given a brief description of this in his book, pages 203 - 205. Mentionable are the attempts of WEBER (1891), KOPEZKY (1899) GEHRHARDT (1901), ROBERTSON (1908), KÖVESSI (1929) and

BACKMANN (1943). We, however, do not claim the history of development of growth functions to be complete.

THOMASIVS (1964) has deduced a growth function similar to that of Richards, otherwise almost everybody has used the growth function developed by Richards since 1959, when the function was developed. In integrated form the function deduced by Thomasius is as follows:

$$W = W_{\max} \cdot \left\{ 1 - e^{-c \cdot T} \cdot (1 - e^{d \cdot T}) \right\}$$

where c and d are growth parameters. Though the function seems suitable for fitting single curves, yet the growth parameters are not so well interpreted as in the function of Richards. We have, therefore, used the growth function of Richards for construction of model for growth curves including the site index curves.

GROWTH MODEL

$$W = A \cdot (1 - b \cdot e^{-k \cdot T})^{\frac{1}{1-m}} \text{ ----- (1)}$$

$$A = a_0 + a_1 \cdot S + a_2 \cdot S^2 + a_3 \cdot S^3 + a_4 \cdot S^4 \text{ ----- (2)}$$

$$b = b_0 + b_1 \cdot S + b_2 \cdot S^2 + b_3 \cdot S^3 + b_4 \cdot S^4 \text{ ----- (3)}$$

$$k = k_0 + k_1 \cdot S + k_2 \cdot S^2 + k_3 \cdot S^3 + k_4 \cdot S^4 \text{ ----- (4)}$$

$$m = m_0 + m_1 \cdot S + m_2 \cdot S^2 + m_3 \cdot S^3 + m_4 \cdot S^4 \text{ ----- (5)}$$

$$S = (H100 - e_0) / e_1 \text{ ----- (6)}$$

In the equations W is the size at time (age) T. In our present studies it is mean stand height or top height or height of single stems, if the method of stem analysis has been adopted for constructing site index curves or mean diameter (breast height, over bark) or basal area of mean tree or living volume (standard or commercial) of mean tree or total volume (living vol. + thinned vol.) per unit area or total number of stems per unit area (or alternatively its natural logarithm) representing a measure of stocking density of a particular evenaged stand. A, b, k & m are the four growth parameters of single growth curves as previously defined and are functions of site quality as above. The relation between site index (H100) and site quality (S) is linear as shown in equation (6) above. The individual growth curve may have no or one point of inflexion lying on any part of the curve, which differs from curve to curve, thus jointly forming a polymorphic curve system. Each growth curve is defined completely by the growth parameters as under:

- (1) The point of inflexion, if occurring, lies at the place where $W = I = A \cdot m^{1/(1-m)}$ where A is the first growth parameter and represents the ultimate limiting value of W when time approaches to infinity, in simple words: when the stem or stand stops putting on increment or dies. The growth curve is asymptotic to the horizontal asymptote (straight line) $W = A$ as shown in figure 1. From above it follows that:

$$\frac{I}{A} = m \left(\frac{1}{1-m} \right)$$

i.e. for a growth curve, if A is constant, then the fourth parameter m is solely responsible for location of the point of inflexion on the curve and it, therefore, exclusively determines the shape (form) of the sigmoid growth curve. Therefore for construction of polymorphic site index curves or other growth curves of similar nature m is the most important parameter and its regression with the site index should closely be studied.

- (2) The second growth parameter b , as already stated, is unimportant biologically and only reflects the choice of zero of time. For site index curves b should always be equal to 1.0 theoretically, since all these curves are bound to pass through origin ($H = 0$, when $T = 0$). For other growth curves b takes different values for different curves, bigger for lower site index and smaller for higher. If the origin is taken when the stem or evenaged stand was planted or started growing, the growth curves do not pass through it. See figure 3.
- (3) The third growth parameter k expresses the rate at which the value of the following linear function of W changes:

$$\log_e \left\{ 1 - (W/A)^{1-m} \right\} = \log_e b - k.T$$

Since the above function of W is specific for each growth curve depending upon m , it is not easy to interpret usefully differences between k 's derived from curves of different forms, although such comparisons have been attempted. Richards infers that the combined value k/m of the third and fourth parameters is the mean relative growth rate of a population wherein all size-classes are represented equally and is also a weighted mean relative growth rate throughout the growth period, the weighing at any time being proportional to the absolute rate at that time. k/m is, moreover, the actual relative growth rate at the point of inflexion on the growth curve. The growth parameter k has thus three different interpretations as mentioned above. The area under the curve is $A^2 \cdot k / (2m+2)$ and is dependant on A , k , and m . The mean height of the curve is, therefore, $A \cdot k / (2m+2)$.

It may be noted here that the growth rate function $dW/dT = n \cdot W^m - k \cdot W$ is totally dependant on the growth function and vice versa. Once the growth function has been fitted to the growth data, the parameters n and K of the growth rate function are automatically known viz.

$$K = k/(1-m) \quad \text{and} \quad n = K \cdot A^{1-m}$$

Lastly it may also be asked whether all the four growth parameters A , b , k and m are really having polynomial (orthogonal) regression with the site quality as shown by us in equations (2) to (5) of the model. We shall come to this point after we have described the methodology of fitting the model to the data.

METHODOLOGY OF FITTING THE MODEL TO DATA

There are three main steps in the methodology of constructing the model for the polymorphic family of site index curves. They are described as under:

Step I - Preliminary analysis of raw data and construction of guiding curve: -

The raw data consists of top height (average crop height previously used) and age figures obtained by measurements of permanent sample plots laid out in normal evenaged stands in various age-classes. In absence of adequate number of permanent sample plots data may be collected from temporary sample plots. The main disadvantage of temporary sample plots is that the plots might not have been normal throughout in the past. The data are also collected during the course of large scale national forest inventories by the standard method of stem analysis or increment borings. Whereas the disadvantages of the data collection by the latter methods are obvious, at times the methods are utilised for economic reasons.

A mean height over age curve is then plotted which is known as the guiding curve. Some authors have impressed that the average height over age curve may not necessarily represent the guiding curve. For instance BRICKELL (1968, on page 8) states that the mean curve does not represent a true height / age curve under the assumptions of equal representation of site quality in all age-classes, rather a curve fitted to medians i.e. 50-percent probability points of the within segment distributions should describe a true height growth curve at that particular level of site index. It is suggested that frequency distributions of the residuals from the guiding curve in each sample segment should closely be studied. If the distribution is not found normal, the studies should further be extended to include computation of indices of skewness and kurtosis; refer KENDALL & STUART (1963). In any case the guiding curve must be constructed with all the precautions and after having the data analysed thoroughly. Heights at different probability levels should

be then computed with the help of the frequency distribution of residuals. Refer BRICKELL (1968), page 9 and 10. The guiding curve along with the other polymorphic curve series of different probability percentage points should be fitted to data according to the procedure given in step II. The authors fully realise the practical difficulties involved in determination of the guiding curve, particularly when we observe many abnormal growth trends in the scatter-diagram. We would only like to emphasise that the statistical methods should be used to determine the exact growth trends. Since we have taken data from various yield tables in present analysis, we have more interest in fitting an appropriate model to the already processed data to generate a system of polymorphic growth curves. We, therefore, proceed to describe the more important step below:

Step II - Use of Richard's growth function to fit individual growth curves to data: -

Richard's growth function can be fitted to the data pertaining to single sigmoid growth curves by many ways. Richards suggests in his paper, pages 292 and 299, that the function developed by him may be fitted to data by the method of STEVENS (1951) as follows:

Growth function-

$$W = A.(1 - b.e^{-k.T})^{1/m}$$

that means $W^{1-m} = A^{1-m} - A^{1-m}. b. e^{-k.T}$

replacing W^{1-m} by y , A^{1-m} by α , $b.A^{1-m}$ by $-\beta$ and $e^{-k.T}$ by ρ^x

we get -

$$y = \alpha + \beta . \rho^x$$

which, as earlier stated, is the equation dealt with by Stevens. A short description of Steven's method may be given here to explain the further procedure. HIORNS (1965) may also be referred.

It is assumed that all observations have been given the same weight. There is no mathematical difficulty in writing down the normal equations to give the least squares or maximum likelihood estimates. Since these equations yield estimates, it will be appropriate to replace α , β and ρ by a , b and r respectively. Note that b here is the estimate of β and is totally different from b , the second parameter of Richard's growth function.

$$\left. \begin{aligned} -a.n & -b. S(r^x) & + Y & = 0 \\ -a. S(r^x) & -b. S(r^{2x}) & + Y_1 & = 0 \\ -a.b.S(r^{x-1}) & -b^2.S(x.r^{2x-1}) & + b.Y_2 & = 0 \end{aligned} \right\} \text{----(i)}$$

where -

$$\begin{aligned} n &= \text{number of observations} \\ S(\dots) &= \text{summation over the } n \text{ values of } x \text{ or } y \\ Y &= S(y) \\ Y_1 &= S(r^x. y) \\ Y_2 &= S(x. r^{x-1}. y) \end{aligned}$$

The information matrix is found by differentiating the expressions in the normal equations with respect to a , b and r in turn, putting each y equal to its expected value and changing all signs:

$$\{I\} = \left\{ \begin{array}{ccc} n & S(r^x) & b.S(x.r^{x-1}) \\ S(r^x) & S(r^{2x}) & b.S(x.r^{2x-1}) \\ b.S(x.r^{x-1}) & b.S(x.r^{2x-1}) & b^2.S(x^2. r^{2x-2}) \end{array} \right\} \text{--(ii)}$$

It will be noted that the first column of terms in the block of normal equations is the first column of terms in the information matrix multiplied by $(-a)$. Similarly the second column is the second column of the information matrix multiplied by $(-b)$. On inverting the information matrix, we find that the covariance matrix has the form:

$$\{V\} = 1/\{I\} = \left\{ \begin{array}{ccc} F_{aa} & F_{ab} & F_{ar}/b \\ F_{ab} & F_{bb} & F_{br}/b \\ F_{ar}/b & F_{br}/b & F_{rr}/b^2 \end{array} \right\} \text{-----(iii)}$$

where F_{aa} , F_{ab} , F_{bb} , F_{ar} and F_{rr} are functions only of r , being in fact the components of the reciprocal matrix:

$$\left\{ \begin{array}{ccc} n & S(r^x) & S(x.r^{x-1}) \\ S(r^x) & S(r^{2x}) & S(x.r^{2x-1}) \\ S(x.r^{x-1}) & S(x.r^{2x-1}) & S(x^2.r^{2x-2}) \end{array} \right\} \text{-----(iv)}$$

Following Fisher's general method, we may start from preliminary inefficient estimates a' , b' , r' and insert these in the left hand side of the normal equations which, instead of yielding exactly zero, now take the small values A , B and R respectively. Efficient estimates a , b and r are now found by adding to the preliminary estimates, the re-

spective increments δa , δb , and δr , where

$$\begin{aligned} \delta a &= F_{aa} \cdot A + F_{ab} \cdot B + F_{ar} \cdot R/b \\ \delta b &= F_{ab} \cdot A + F_{bb} \cdot B + F_{br} \cdot R/b \quad \text{-----(v)} \\ b \cdot \delta r &= F_{ar} \cdot A + F_{br} \cdot B + F_{rr} \cdot R/b \end{aligned}$$

In consequence of the relations, noted above, between columns of the block of normal equations and columns of the information matrix, the expressions for the increments simplify to:

$$\begin{aligned} \delta a &= -a' + F_{aa} \cdot Y + F_{ab} \cdot Y_1 + F_{ar} \cdot Y_2 \\ \delta b &= -b' + F_{ab} \cdot Y + F_{bb} \cdot Y_1 + F_{br} \cdot Y_2 \quad \text{----(vi)} \\ b \cdot \delta r &= F_{ar} \cdot Y + F_{br} \cdot Y_1 + F_{rr} \cdot Y_2 \end{aligned}$$

Hence efficient estimates are:

$$\begin{aligned} a &= a' + \delta a = F_{aa} \cdot Y + F_{ab} \cdot Y_1 + F_{ar} \cdot Y_2 \\ b &= b' + \delta b = F_{ab} \cdot Y + F_{bb} \cdot Y_1 + F_{br} \cdot Y_2 \quad \text{---(vii)} \\ r &= r' + \delta r \end{aligned}$$

where

$$\delta r = (F_{ar} \cdot Y + F_{br} \cdot Y_1 + F_{rr} \cdot Y_2) / b$$

It may be noted that the preliminary estimates of α and β have fallen out of the equations. This means that we need find a preliminary estimate only for ρ : the estimates of the other two parameters are then given explicitly in terms of functions which depend only on the preliminary estimate of ρ and, of course, on the observations.

Calculation of standard errors:

Finally we have to determine the sum of deviations from the fitted regression curve:

$$S(y - y_e)^2 = \text{sum of squared deviations}$$

The formulae for the standard errors of the estimate are stated as under:

$$\left. \begin{aligned} S.E.(a) &= \sqrt{(F_{aa} \cdot s^2)} \\ S.E.(b) &= \sqrt{(F_{bb} \cdot s^2)} \\ S.E.(r) &= (\sqrt{(F_{rr} \cdot s^2)}) / b \end{aligned} \right\} \text{-----(viii)}$$

where

$$s^2 = S(y - y_e)^2 / (n - 3)$$

Equally spaced ordinates:

Although in the above procedure an arbitrary series of values of x can be used, the most simple, natural and useful series is one in which the values of x advance in equal steps. If there are n values of x , we may, by a simple change of units and origin, designate these as:

$$x = 0, 1, 2, \dots (n-1)$$

Stevens has prepared tables of the components of the F matrix for $n = 5, 6$ and 7 . These tables have now been extended by HIORNS (1965) for $n = 8, 9, \dots, 30$, thus increasing the utility of the standard method of Stevens.

It may be noted that:

- (i) The growth equation of Richards becomes insoluble when $m = 1$, nevertheless a Gompertz curve* can still be fitted with considerable accuracy by Stevens's method if a value of, say 0.999 or 1.001 be chosen for m .
- (ii) An iterative process is necessary for determining the value of m . The number of iterations is, however, very low, as the preliminary estimate of m can be found out with fair amount of accuracy by the method given by RICHARDS (1959, page 295) by plotting the growth rate dW/dT over the value W/A . It is quite easy to make an initial guess of parameter A by simply plotting the data on a graph paper. Also refer RAWAT (1973) for this, who has estimated the value of m by Richards's method for a number of species.

HIORNS (1965) describes the method of Stevens and other methods available for asymptotic regression and claims that in general none is better than Stevens's method. He quotes some methods due to HARTLEY (1948), PATTERSON (1956, 1958, 1960), PATTERSON & LIPTON (1959) and GOMES (1953). With these methods there is loss of efficiency and estimates may be biased; these features have been discussed by FINNEY (1958) and HARTLEY (1959). NELDER (1961) has worked further on logistic curves of BERKSON (1953). TURNER, MUNROE & LUCAS (1961) can also be said to have dealt with the problem. Probit analysis developed by BLISS (1935) and extended further by FINNEY (1952) is also an important step towards analysis of so-called sigmoid response curves. Hiorns, however, misses the name of MARQUARDT (1963). Mar-

*Gompertz curve is $W = A \cdot e^{-b} \cdot e^{-k \cdot T}$

$$\text{or } \log_e W = \log_e A - b \cdot e^{-k \cdot T}$$

Replacing $\log_e W$ by y , $\log_e A$ by α , b by $-\beta$ and $-k \cdot T$ by $x \cdot \log_e \rho$,

we get -

$$y = \alpha + \beta \cdot \rho^x \quad \dots \text{ (Stevens's equation)}$$

Marquardt's method of maximum neighbourhood is an improvement over the Gauss-Newton method. The method is based on the expansion of a non-linear function to a Taylor series combined with the method of steepest descent or its various modifications. The method is suitable for any non-linear function with any number of parameters and is available for processing in electronic computers as IBM Share Program No. 1428, FORTRAN program. The computer program is now available in computer language PL/I also as Share Program SDA 3094.01, NLIN-2.

In forest science models for site index curves have so far been constructed by BRICKELL (1966, 1968), LUNDGREN & DOLID (1970), BECK (1971), HÄGGLUND (1972), RAWAT (1973) and probably by some other authors. All of them have used the growth function of Richards. Whereas LUNDGREN & DOLID (1970) and RAWAT (1973) took data from existing yield tables, the rest of the authors did the work from raw data. TURNBULL & PIENAAR (1965) also used the growth function to construct a model for analysis and prediction of growth in non-normal and thinned forest stands. Out of these authors none except RAWAT (1973) fitted the function by method of Stevens, the suitability of which has already been discussed. Details of the computer program for Stevens's method of curve fitting are available with the authors. It may be also mentioned here that the method is much more simple than the others, in that the facilities of electronic computer for it are desirable but not necessary. Further HÄGGLUND (1972) remarks on page 256 that the Gauss-Newton method does not give a certain convergence, which is not the case with Stevens's method. NELDER (1961) comments on Stevens's method as under:

"Several papers concerned with fitting of the curve

$z = \alpha + \beta \cdot \rho^t$ have appeared, but these are almost entirely concerned with the case where z has constant variance and where t_i are equally spaced. Where time scale based on meteorological measurements (such as day-degrees) are used for field crops, it is usually impossible or impracticable to arrange for equally spaced t 's, so that methods used in the abovementioned papers are no longer available, even if it could be assumed that $\text{var}(z)$ was constant."

We agree with Nelder as we require equally spaced values of t (time) in Stevens's method, if we like to utilise the facilities of the tables for calculating the covariance matrix. This is not required in some other methods. But for our purpose Stevens's method is still suitable as we take measurements of permanent sample plots or in stem analysis at equal intervals of 5 or 10 years. Particularly the tables for calculating the covariance matrix are available upto $n = 30$ and now it becomes very easy to reduce the time scale to suit our purpose. If this is not possible under exceptional circumstances, the method developed by Marquardt or by

others are still available, but then the facilities of electronic computer are unavoidably required. We, therefore, prefer the accurate and standardised method of Stevens for fitting Richards's growth function. Moreover standard errors of parameters can be calculated in the method and thus their upper and lower limits can be predicted.

Step III - Completion of model construction work by harmonising the polymorphic family of growth curves: -

After having fitted the function to the data for individual growth curves the next step is to study regression between various growth parameters and site quality, so that a mathematical model may be constructed to generate a system of polymorphic growth curves. Regression of growth parameters A and b with the site index is simple and poses no problem. It may be made clear here that site quality (and not the site index) has been taken by us to study the regression, as site quality is in complete numbers and facilitates the use of orthogonal polynomials, which are much more simple than the general polynomials and can be fitted even with the help of already available tables; see FISHER & YATES (1957) and SPRENT (1969, page 74). Site quality and site index are having perfect linear regression and no accuracy is lost in the process. This has at the same time helped us in simplifying the regression studies. In most of the cases we don't require a polynomial higher than the second order (parabola) and in one or two cases only the first order (straight line) polynomial was enough. Usual F-test was applied in studying the regression. Standard IBM-Program (1966) was applied for the purpose. We have studied the regression for ten tree species, seven of them growing in Germany viz. Spruce, Pine, Fir, Larch, Douglas Fir, Beech and Oak and rest three growing in India viz. Teak, sal and Deodara. In this paper for want of space results are given for three species only, namely for Spruce, Pine and Deodara. Also see fig.2. The regression studies of these two growth parameters (A & b) with site quality (site index) are in line with those made by other authors.

The most important regression studies are of growth parameters m & k with site quality. BRICKELL (1968) could not find any regression between site index and m and had, therefore, to keep m constant for all levels of site index, thus restricting the true polymorphic nature of the growth curves. LUNDGREN & DOLID (1970) have kept k & m both parameters constant for all site index levels and thus lost the flexibility of the function. BECK (1971) states that keeping m constant for all site index levels no significant difference was observed in standard error of estimate. It is most probably on account of the fact that he used Marquardt's method of curve fitting, in which all the growth parameters can be simultaneously changed to get the best fit. We are still not definite if Beck could get suitable regression between m and site index by determining m first by the method of Richards and then using

the method of Stevens. HÄGGLUND (1972) on page 256, who also used the Newton-Gauss method (improved by Marquardt) got a satisfactory regression of the growth parameters m and k with A as under:

$$k = a + b \cdot A^c \quad \text{and} \quad m = d + e \cdot A^f$$

He thus replaced site index by A . We do not know how we are benefitted by that. Though it is not easy to find out suitable regression of m and k with the site index, we found that a polynomial regression upto the fourth order is generally sufficient to maintain the desired accuracy (1 foot for top height). The site index curves may be constructed right from zero age (beginning) and there is no hesitation in doing so, as it existed before. Still we would like to confess that regression of m with site index is, though simplified by fitting the curve by method of Stevens, yet not suitably represented by polynomial regression.

NON-LINEAR ASYMPTOTIC REGRESSION STUDIES FOR OTHER GROWTH CURVES AND SUITABILITY OF THE MODEL FOR STOCKING DENSITY OF EVENAGED STANDS

After having described the methodology of fitting asymptotic curves to growth data of site index, we would also like to discuss here the properties of other growth curves viz.

- (i) Diameter (b.h.o.b)/age curves for single stems or mean tree of stand.
- (ii) Basal area/age curves for single stems or mean tree of the stand.
- (iii) Living volume (standard or commercial)/age curves for single stems or mean tree of stand.
- (iv) Total volume/age curves for per unit area of stand. Total vol. = Living vol. + Thinned vol.
- (v) Number of stems per unit area/age curves of stand for a given stocking density.

When we say mean tree of stand, we presume that the even-aged stand has been maintained at a particular stocking density level throughout its life through a continuous thinning-policy with equally spaced thinning cycles and a constant thinning intensity. Refer to JOHNSTON & BRADLEY (1963).

SPURR (1952) pages 211 - 216 describes the properties of first three of the above curves. The fourth curve is an ideal case for non-linear asymptotic studies. The fifth curve is not a growth curve but is intimately related to growth. All these five curves have basically the same properties as site index curves have. The point of inflexion, if at all occurring, lies at different places on the curves depending upon the stocking density, site and environment and leaving apart the genetic factor. The fifth curve, representing the stocking density by number of stems per unit area of an evenaged stand for a parti-

cular grade of thinning is of particular importance and needs a bit more attention. It appears that few research workers have tried to study its true non-linear asymptotic nature with age, having one point of inflexion in a very early age. The number of stems per unit area at a particular age varies from species to species, site to site and from one plantation technique or management practice to other. Generally the number is quite large in the beginning and it does not interest the researchers till thinning operations are started. We found that if common logarithm of number of stems per unit area is plotted over age, the resulting curve can be represented well by the model proposed by us. The asymptote $W = A$ falls below the curve as against in growth curves, where the asymptote lies above the curves. TURNBULL & PIENAAR (1965) have proposed a mathematical system of basal area/acre over age curves for a given site quality to define the expected trend of growth for stands of different stocking levels. The scheme of correlated curved trend (C.C.T.) plots, originally proposed by O'CONNOR (1935) and worked upon by MARSH (1957), PIENAAR (1965) and others seems to be a good approach to problems in thinning research. There is also no doubt that basal area is the simplest and most widely accepted measure of stocking density. However while examining the mathematical system and working it out for own species we noticed that stand basal area per acre/age curves are not asymptotic in most of the cases as advocated by them. These curves bend downwards after reaching the maximum value for stands of some of the species like Pine (*Pinus silvestris*, L.). The data has been taken from the yield tables of WIEDEMANN (1957). See fig. 4. It is because of the fact that the site is not capable of supporting optimum number of stems per unit area at higher ages. (See ASSMANN & FRANZ (1972)). When this downward trend starts we start losing optimum volume, but not necessarily money as the market trend may be much more positive for higher dimension of timber. It is this analysis that lead us to think on other lines. Though the subject pertains to thinning research requiring detailed investigations and analysis, we found that stand basal area represented by mean tree over age (growth curve (ii)) and the number of stems per unit area over age (curve (v)), when combined, are better suited to define the different stocking density levels of stands as the base of the proposed model. At the same time both types of curves maintain asymptotic character with age and the model proposed by us is well suited to both of them. See fig. 5*6. THOMAS (1964) uses the so-called KORSUN's function (1935), which is a double logarithmic parabola, for the number of stems per hectare over age relationship. The function does not have enough flexibility to represent the biological growth trends with desirable accuracy.

The model can be fitted to total volume per unit area over age curves (curve (iv)) similarly and subtracting from it the product of growth curve (iii) and (v) for a particular age would give us thinned volume for that age, for which suitable regression curves were not available uptill now. For want of space it is not pos-

sible to give here all the results. However diameter over age curves (curve (1)) are reproduced in fig. 3.

A CRITICAL LOOK AT THE MODEL

The model, no doubt, is fairly accurate for all the growth curves and is proposed to be used widely for yield and growth studies. We offer here a concept of computerised yield tables of evenaged stands, based entirely on the model. Facilities of electronic computer are desirable to construct the model but are not necessary if standard method of Stevens is used to fit Richards growth function to individual growth curves. A further critical look, however, would show that:

(i) The non-linear function can not be transformed into a linear form except for a particular level of site index. Refer TURNBULL & PIENAAR (1965, page 3). The model, therefore, has all the disadvantages of a non-linear system. The methods of linear regression are not available for it.

Thus it becomes a bit difficult and beyond the reach of an average practising forester.

(ii) The function is nonconvertible i.e. its equation can not be readily solved for site index, when age and top height are known. On the other hand if site index is known, we can calculate top height for a given age and vice versa. Thus site index can only be estimated by interpolations between successive tables or curves. Interpolation in a non-linear system can lead to inaccuracies. Therefore we have to look for some alternative, by which site index can be calculated directly when top height and age are known.

Attempts have been made by some authors in this direction. BRICKELL (1970) and HÄGGLUND (1972) have listed most of the approximate functions for the purpose and found that different approximate functions are suitable for different tree species. No general choice of a function can be made over the rest. The following approximate functions have been found suitable by us:

$$(1) \quad H_{100} = (H/e^{-a/T})^{1/(1+e^{-b/T})}$$

$$(2) \quad H_{100} = H + b_1 \cdot (\log T - \log 100) + b_2 \cdot (T^2 - 100^2) + b_3 \cdot \left(\frac{H}{T} - \frac{H}{100}\right) + b_4 \cdot \left(\frac{H}{T^2} - \frac{H}{100^2}\right)$$

$$(3) \quad H_{100} = H \cdot \left(1 + a \cdot \left(\frac{1}{T} - \frac{1}{100}\right)\right)$$

where H_{100} is site index and H is top height at age T .

CONCLUSIONS

Detailed non-linear asymptotic regression studies have been carried out on tree and stand growth for ten economically important tree species occurring in Germany and India. A mathematical (growth) model has been constructed for all the growth curves in general and for the site index curves in particular. The system of growth curves generated by the model satisfies general principles of growth process of trees. Polymorphic system of curves results when Richards's growth function is used for curve fitting. It is proposed that the function be fitted to growth data by standard method of Stevens. The method facilitates manual handling of data, although facilities of electronic computer are desirable. Standard errors of growth parameters can also be calculated if Stevens's method is used for the purpose.

Site quality is proposed to be used instead of site index to study its regression with the growth parameters, because it facilitates the use of orthogonal polynomial tables, thereby reducing the computational work.

The model is flexible and is, therefore, equally suitable for all other growth curves. The model can also be used for defining stocking density over age relationship for different thinning regimes, when the following two curves are combined:

- (i) Stand basal area represented by mean tree of the stand over age curve under the prescribed thinning regime.
- (ii) Number of stems per unit area over age curve under the same thinning regime as in (i) above. The curve is better defined when common logarithm of number of stems is used.

The proposed model may be utilised for analysis and interpretation of data collected for thinning research.

The growth function is mathematically inconvertible. Site index can not be directly calculated by it, if top height and age are known. Site index may, therefore, be calculated by some suitable approximate functions proposed in the paper.

ACKNOWLEDGEMENT

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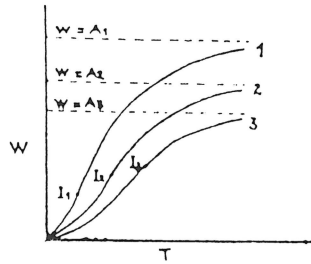


Fig. 1(a)

Fig. 1(a) - shows Richards's growth function fitted to three polymorphic growth curves. Their three different horizontal asymptotes and points of inflexion have also been shown.

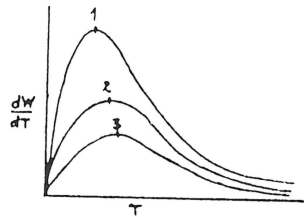


Fig. 1(b)

Fig. 1(b) - shows Richards's growth rate function for three different growth rate curves. The function is able to represent the differences among various ages of culmination varying with site.

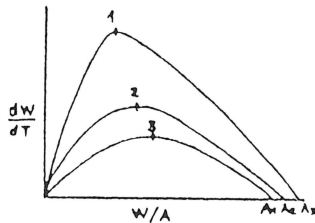


Fig. 1(c)

Fig. 1(c) - shows the relationship between growth rate dW/dT and size W , when W is measured in terms of its ultimate asymptotic value A . The position of maximas of different curves gives us preliminary estimate of m , the fourth growth parameter of Richards's growth function.

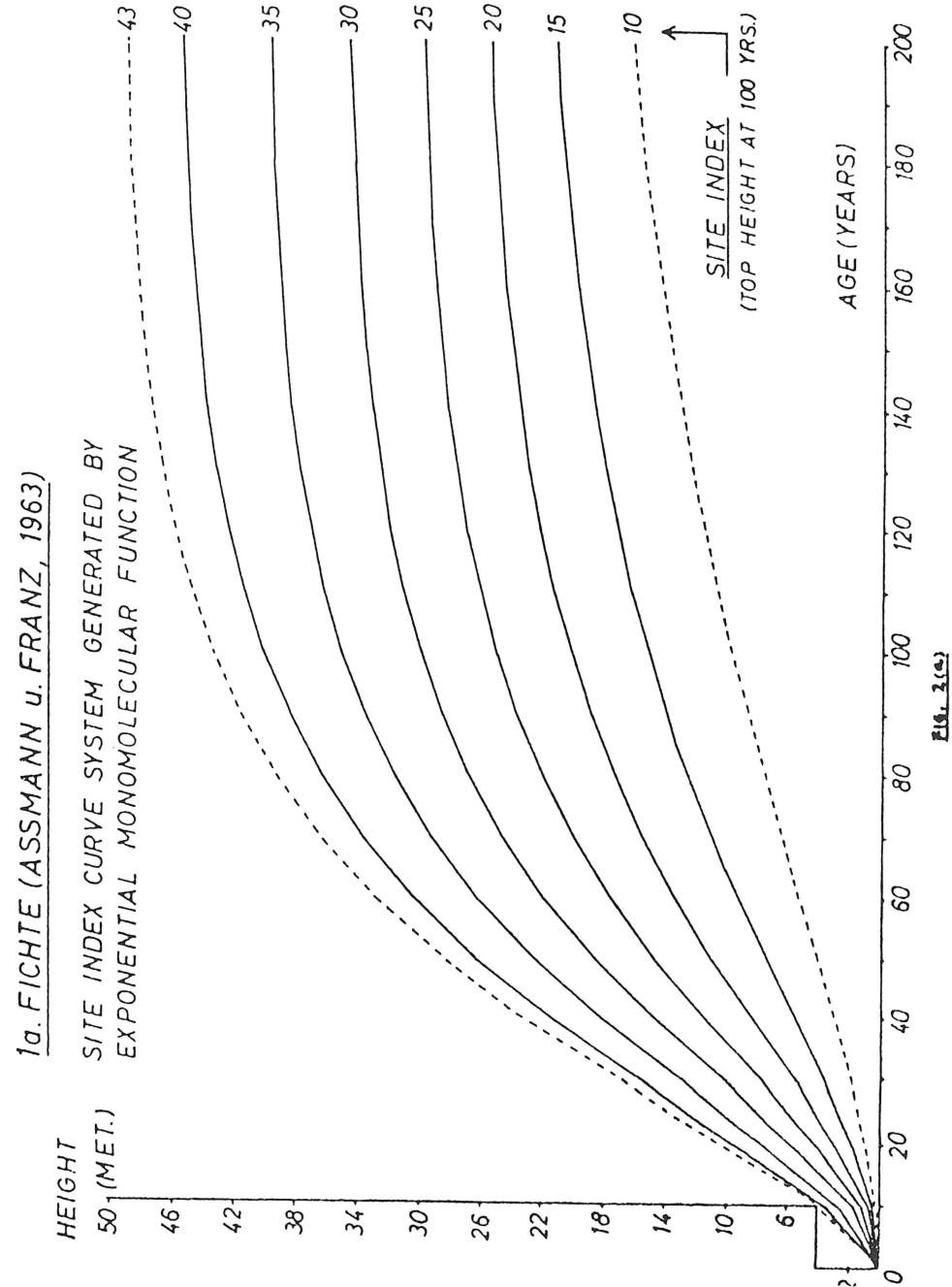


Fig. 2(c)

2. KIEFER

SITE INDEX CURVE SYSTEM GENERATED BY THE EXPONENTIAL MONOMOLECULAR FUNCTION

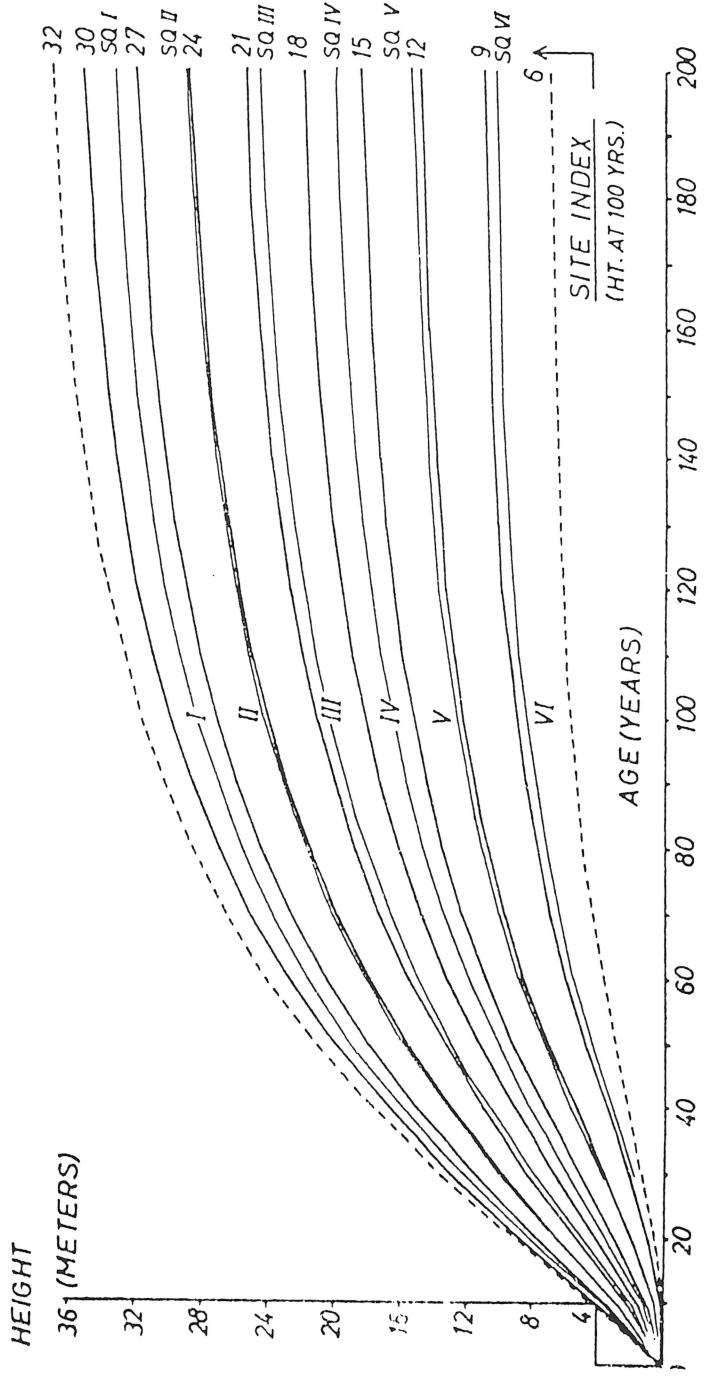
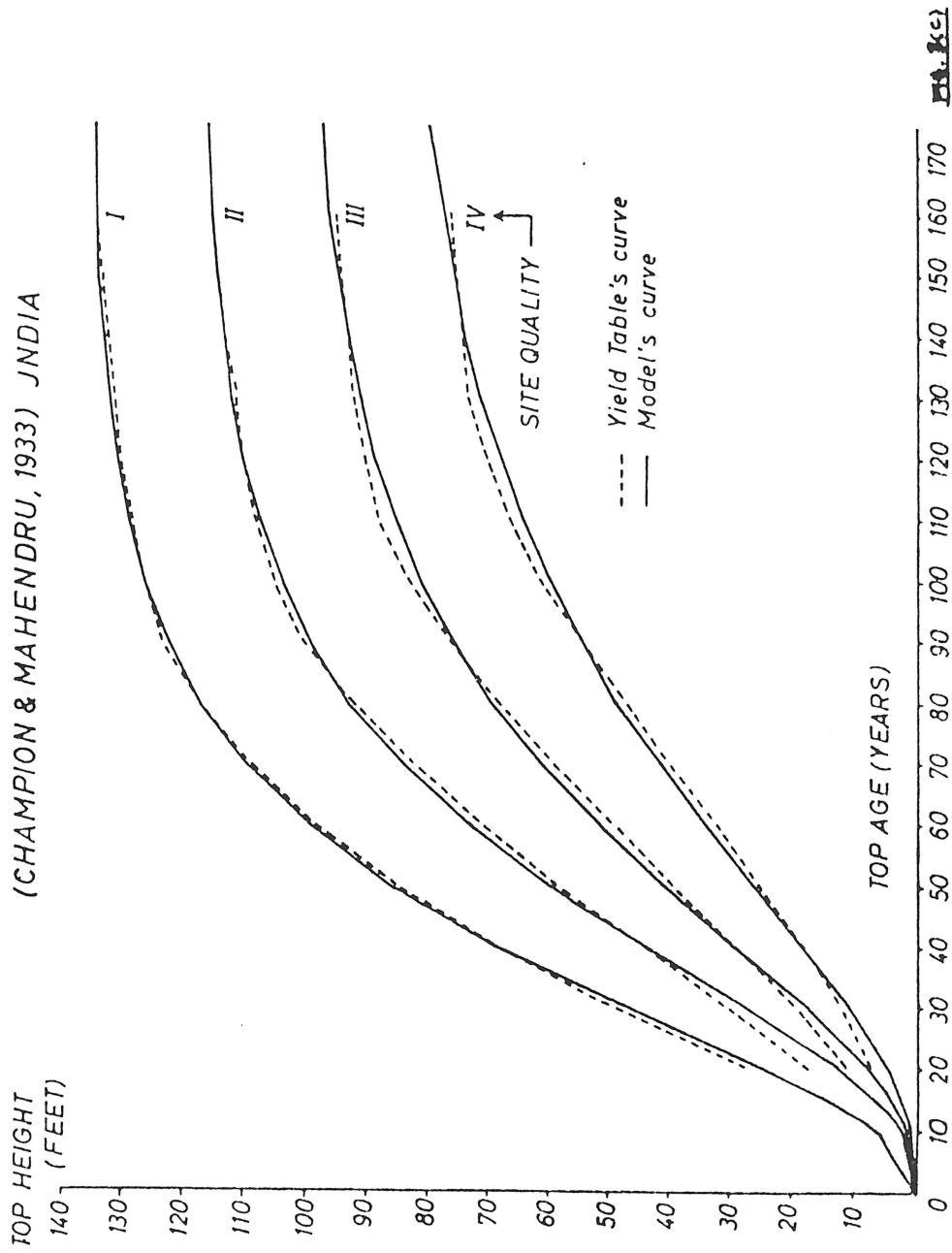


Fig. 2 (b)

CEDRUS DEODARA (CHAMPION & MAHENDRU, 1933) JNDIA



SPRUCE (FICHTE, ASSMANN-FRANZ, 1963)

GERMANY, Diameter/Age Curves

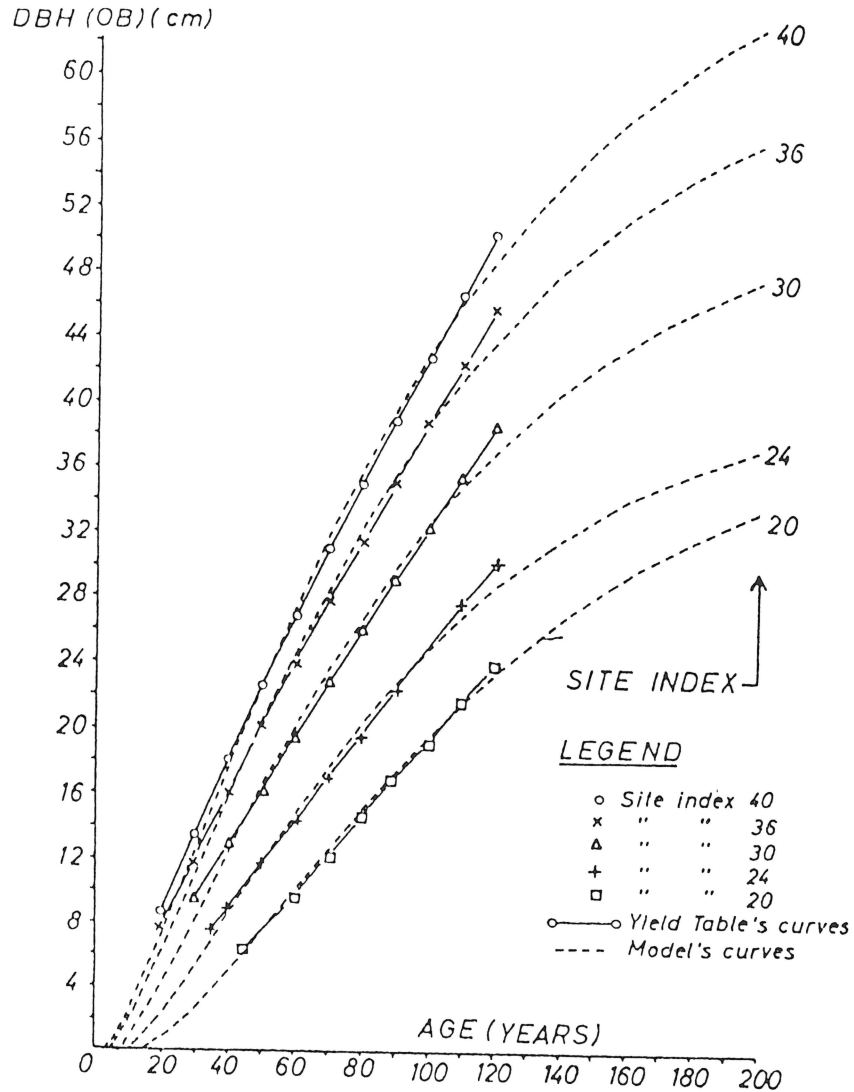


Fig. 3(a)

PINE (KIEFER - WIEDEMANN, 1943)

GERMANY

DIAMETER/AGE CURVES

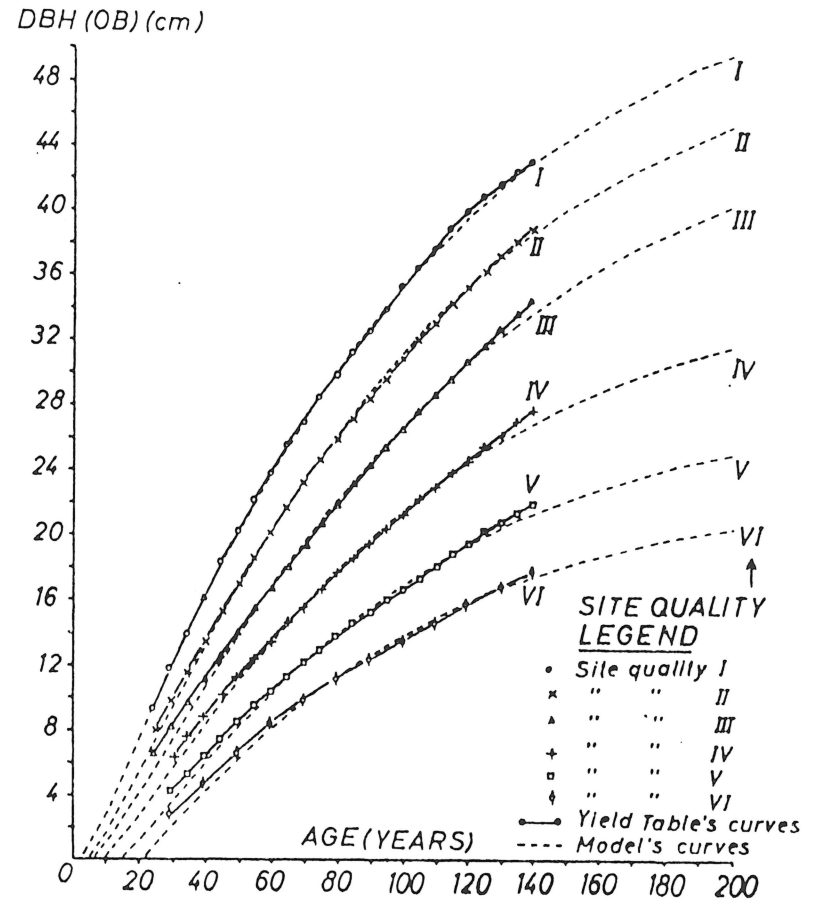


Fig. 3(b)

DEODARA (CHAMPION & MAHENDRU, 1933)
JNDIA

DIAMETER / AGE CURVES

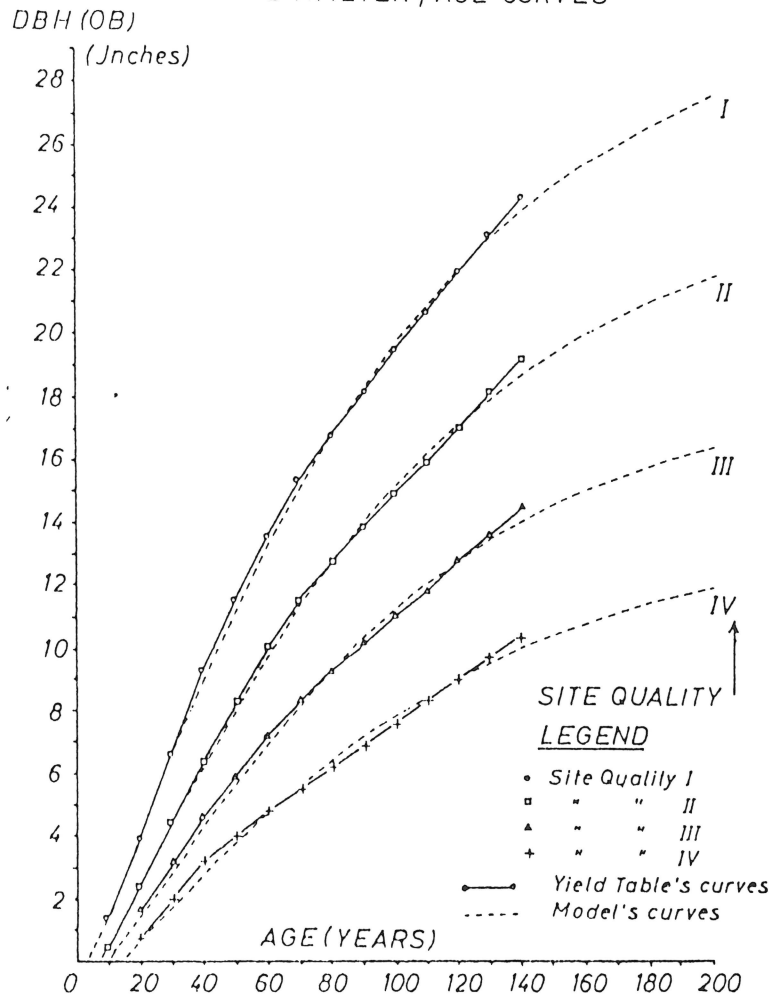


Fig. 3(o)

SPRUCE (FICHTE, ASSMANN-FRANZ 1963), GERMANY

Basal area per hectare / age curves
Middle yield-level, optimum stocking density

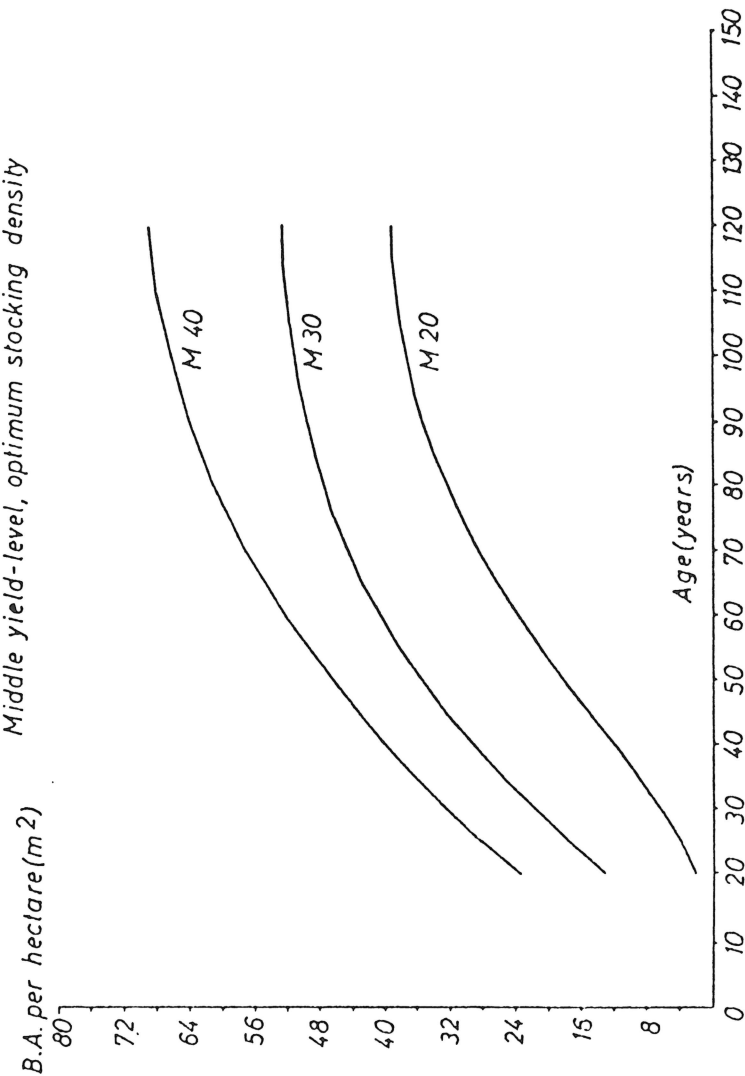


Fig. 4(a)

PINE (KIEFER, WIEDEMANN -1943) GERMANY

Basal area per hectare / age curves

C grade thinnings

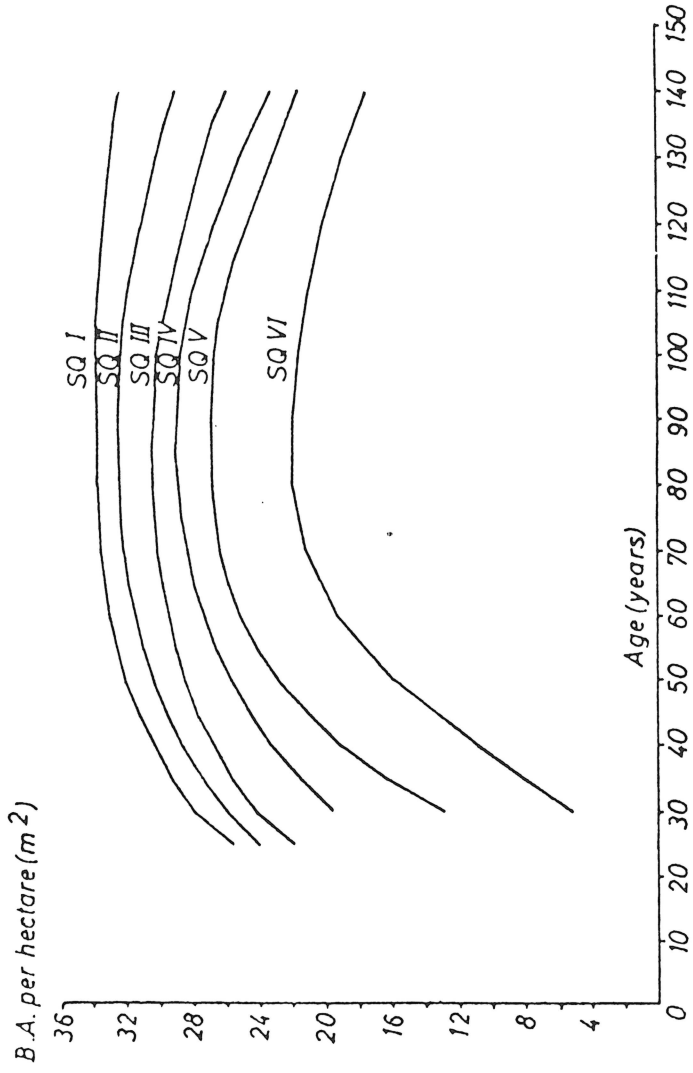


Fig. 4(b)

CEDRUS DEODARA (CHAMPION & MAHENDRU, 1933)

JNDIA

B.A. per acre (ft²)

Basal area per acre / age curves

C grade thinnings

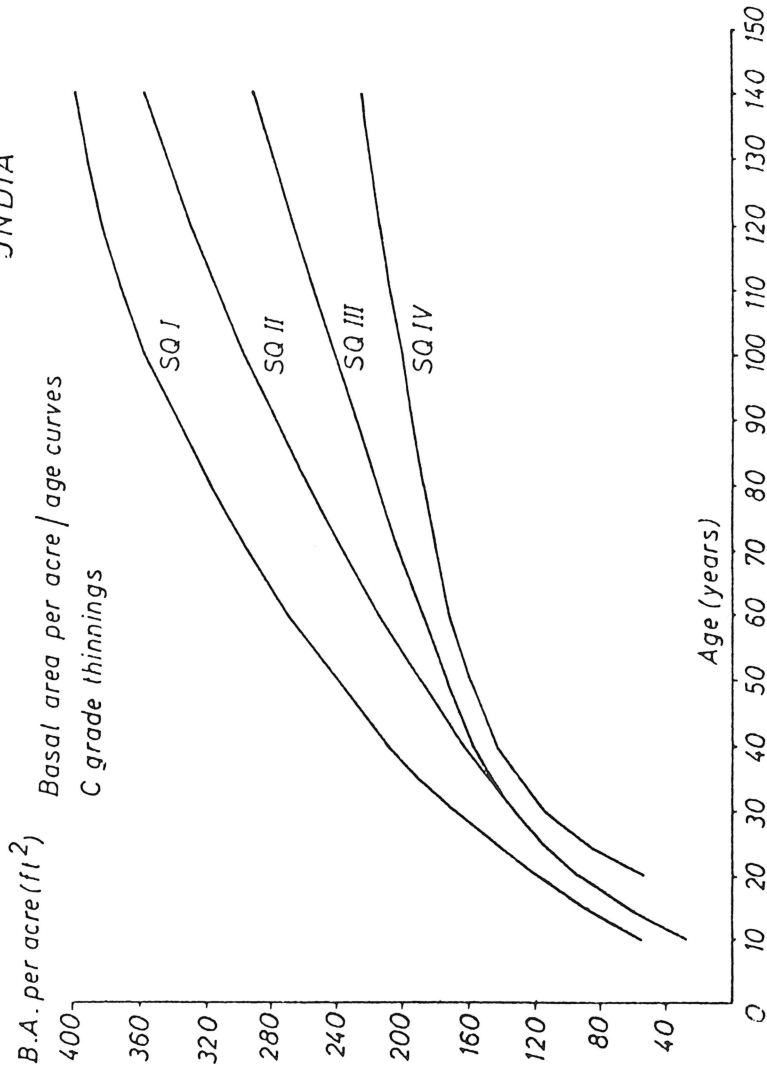


Fig. 4(c)

SPRUCE (FICHTE, ASSMANN-FRANZ, 1963) GERMANY

Basal area of mean tree / age curves
Middle yield-level, optimum stocking density

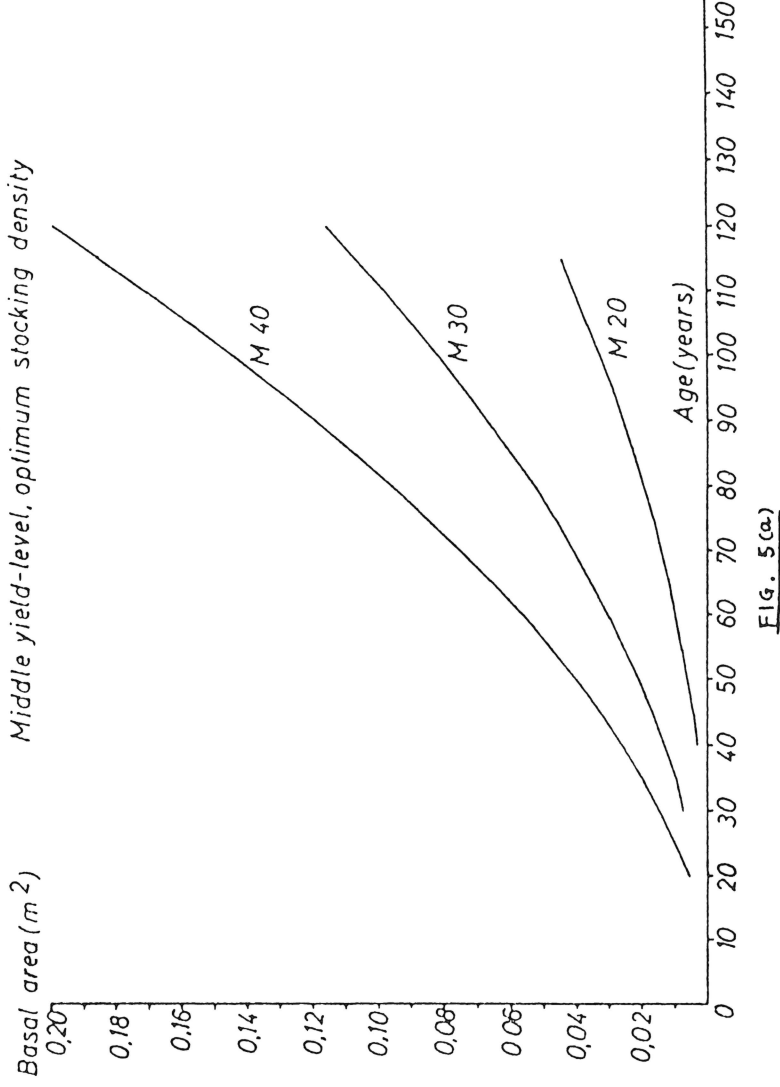
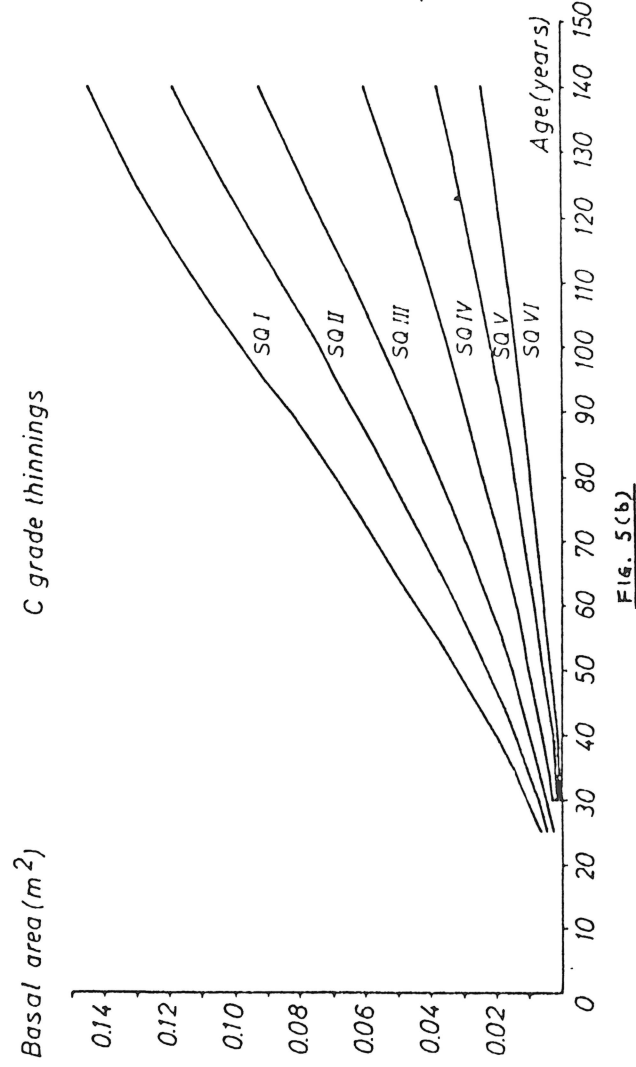


Fig. 5(a)

PINE (KIEFER, WIEDEMANN - 1943) GERMANY

Basal area of mean tree / curves



C grade thinnings

Fig. 5(b)

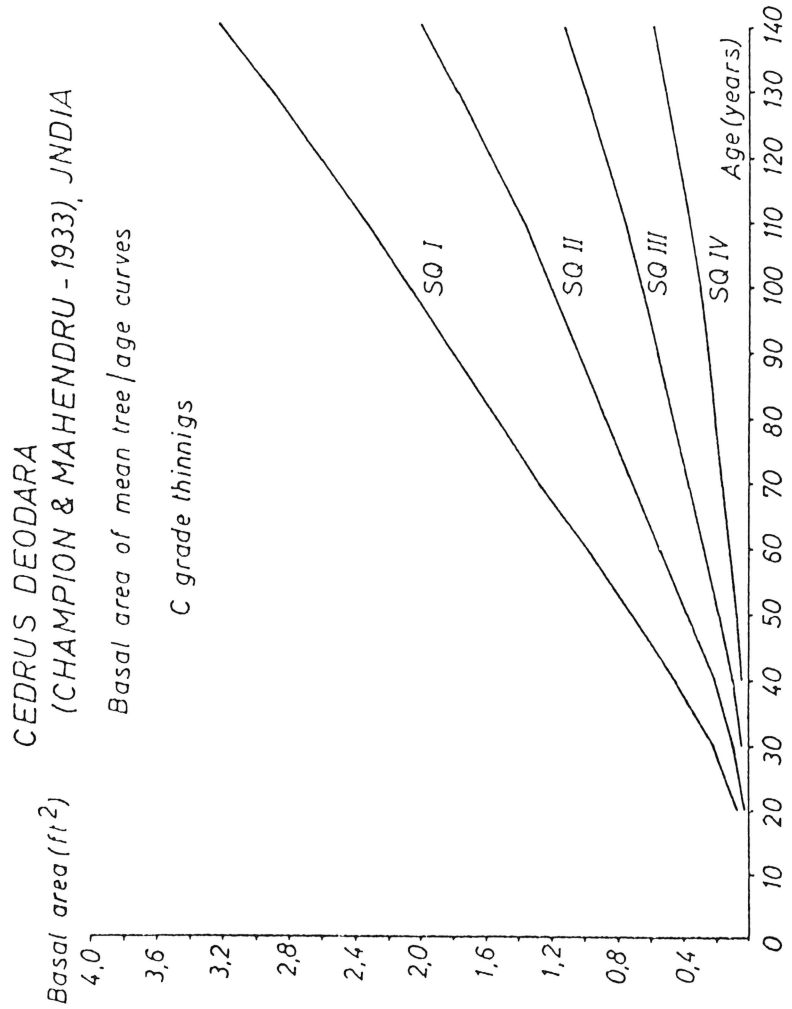


Fig. 5(c)

SPRUCE(FICHTE, ASSMANN-FRANZ 1963) GERMANY

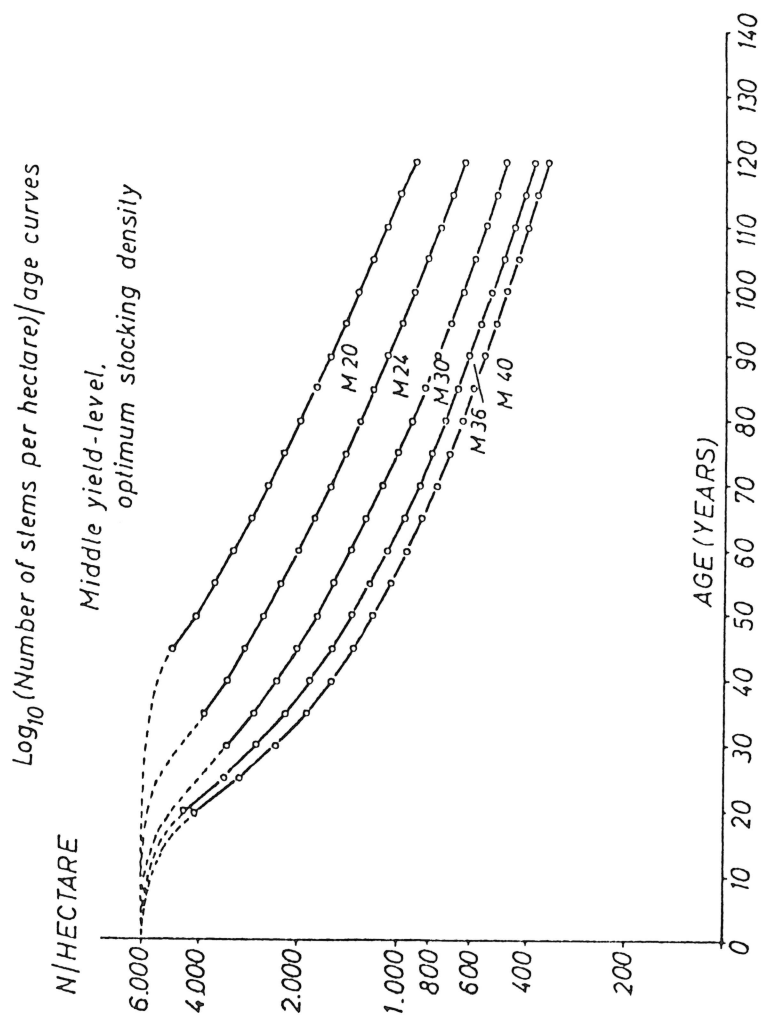


Fig. 6 (ca)

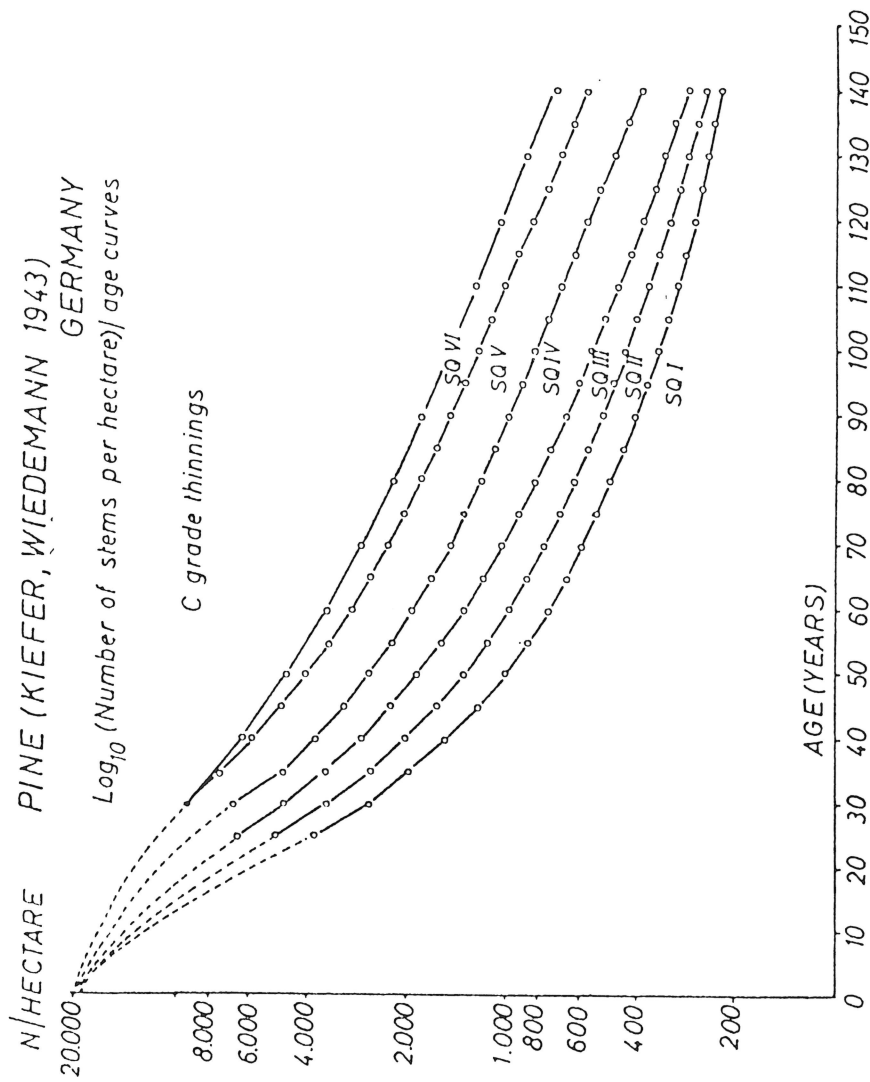


FIG. 6 (b)

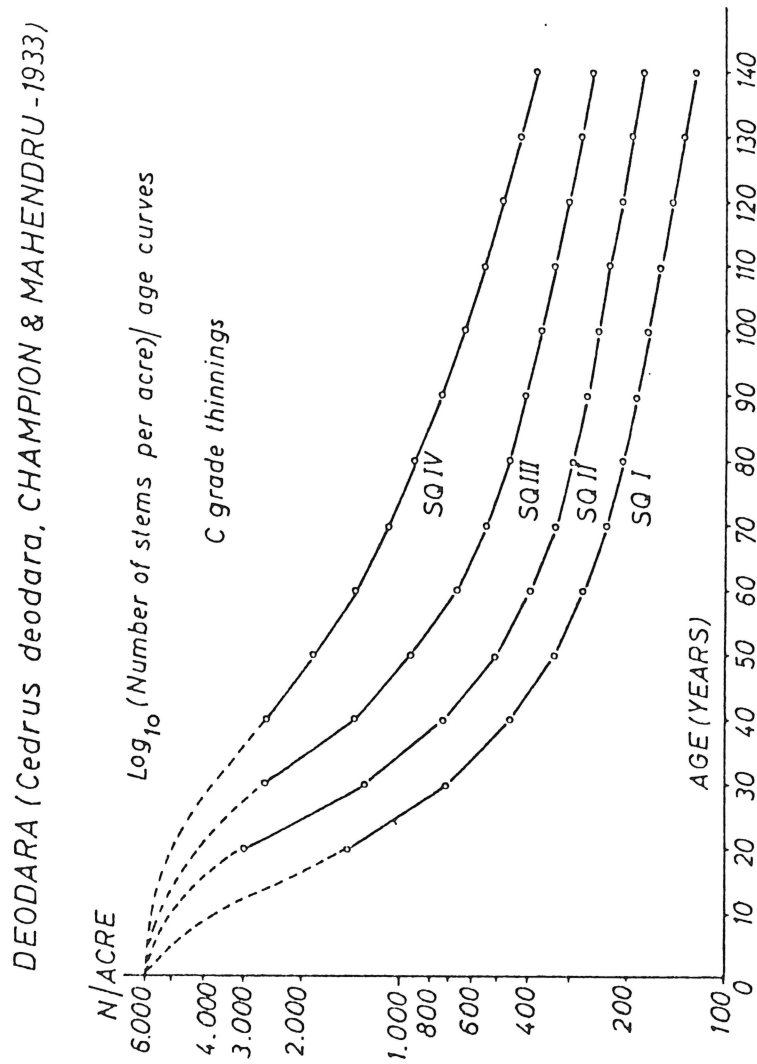


FIG. 6 (c)

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Edited by Jöran Fries*



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INTRODUCTION

The development of growth models has been an important part of the research in growth and yield since the end of the last century. Models were originally based on graphical descriptions of the development of stand parameters such as mean height, basal area, volume, mean diameter, etc. Later, mathematical descriptions of growth processes were also used.

A new era began when regression analysis was introduced into growth and yield studies in the 1930s. During the past decades, a number of new methods in multi-variate analysis have been successfully applied to growth and yield problems. Many of these methods involve elaborate calculations, which can hardly be performed without the fast computers now available. These computers have also made it possible to handle large amounts of data and thus to develop and apply growth models not only for stands but also for single trees.

As a result of this recent rapid progress, there is today, all over the world, much activity in developing growth models. The subject "growth models for tree and stand simulation" was therefore chosen as the main theme of two meetings in 1973 of IUFRO Subject Group S4.01 Mensuration, Growth and Yield. The papers presented at these two meetings form the contents of this book.

I wish to express my gratitude to all those who helped in organizing the two meetings. I should especially like to mention Jean Pardé and William Warren, who were responsible for organizing the meetings in Nancy, France and Vancouver, British Columbia, Canada, respectively, and Noël Decourt and Terry Honer, who took a large part in the collection of the papers.

Jöran Fries
Leader of IUFRO S4.01

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